MATH 2020 Advanced Calculus II

Tutorial 5

1. Rewrite each of the following integrals by using the given substitution. For Q1, compute also the integral.

   (a) $\iiint_S (x^2+y^2)dV$ where $S = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1 \right\}$. $(x = au,\ y = bv,\ z = cw)$

   (b) $\iint_R \left(\left(\frac{y}{x}\right)^2 + (xy)^2 \right) dA$ where R is bounded by $y = 4x,\ y = 3x,\ xy = 2$ and $xy = 1$. ((i) $p = xy,\ q = \frac{y}{x}$; (ii) $x = \frac{u}{v},\ y = uv$)

   (c) $\iint_R dA$ where R is bounded by $xy = 0,\ xy = 1,\ y = 2$ and $y = 3$. $(u = y,\ v = xy)$

   (d) $\int_0^1 \int_0^{\sqrt{1-x}} (x^2+y^2)dydx$. $(x = u^2 - v^2,\ y = uv)$

   Solution.

   (a) The Jacobian is

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

   The solid S is transformed onto the unit ball $\{u^2+v^2+w^2 \leqslant 1\}$. The integral thus becomes

$$\begin{aligned}
&\iiint_{u^2+v^2+w^2\leqslant 1} [(au)^2 + (bv)^2](abc)dudvdw \\
&= abc \int_0^{2\pi} \int_0^{\pi} \int_0^1 [a^2(\rho\sin\phi\cos\theta)^2 + b^2(\rho\sin\phi\sin\theta)^2]\rho^2 \sin\phi\ d\rho d\phi d\theta \\
&= abc \left(\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)d\theta \right) \left(\int_0^{\pi} \sin^3\phi d\phi \right) \left(\int_0^1 \rho^4 d\rho \right) \\
&= abc \times \pi(a^2+b^2) \times \frac{4}{3} \times \frac{1}{5} \\
&= \frac{4\pi}{15} abc(a^2+b^2).
\end{aligned}$$

   Remark. The moment of inertia $I_z(S)$ of the ellipsoid S with respect to the z-axis is defined to be

$$I_z(S) = \iiint_S (x^2+y^2)\delta_S\ dV.$$

   If the density δ_S is constant, then we have

$$I_z(S) = \delta_S \times \frac{4\pi}{15}abc(a^2+b^2) = \frac{m}{\frac{4\pi}{3}abc} \times \frac{4\pi}{15}abc(a^2+b^2) = \frac{m}{5}(a^2+b^2).$$

(b) (i) The Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(p,q)}\right| = \left|\frac{\partial(p,q)}{\partial(x,y)}\right|^{-1} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}^{-1} = \left(\frac{2y}{x}\right)^{-1} = (2q)^{-1}.$$

The region R is transformed onto the rectangle $\{1 \leqslant p \leqslant 2,\ 3 \leqslant q \leqslant 4\}$. The integral thus becomes

$$\int_3^4 \int_1^2 (q^2 + p^2)(2q)^{-1} dp dq.$$

(ii) The Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

The region R is transformed onto the rectangle $\{\sqrt{1} \leqslant u \leqslant \sqrt{2},\ \sqrt{3} \leqslant v \leqslant \sqrt{4}\}$. The integral thus becomes

$$\int_{\sqrt{3}}^{\sqrt{4}} \int_{\sqrt{1}}^{\sqrt{2}} [(v^2)^2 + (u^2)^2] \left(\frac{2u}{v}\right) du dv.$$

(c) The Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \begin{vmatrix} 0 & 1 \\ y & x \end{vmatrix}^{-1} = |-y|^{-1} = u^{-1}.$$

The region R is transformed onto the rectangle $\{2 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 1\}$. The integral thus becomes

$$\int_2^3 \int_0^1 u^{-1} dv du.$$

(d) The Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2(u^2 + v^2).$$

Let $\Phi(u,v) = (u^2 - v^2, uv)$. Let T_1 be the triangle with vertices $(0,0),\ (1,0)$ and $(1,1)$. Let $T_2 = -T_1$ be the triangle with vertices $(0,0),\ (-1,0)$ and $(-1,-1)$. Then Φ maps diffeomorphically T_1 (resp. T_2) onto R (interior onto interior, boundary onto boundary). We need to choose either T_1 or T_2 for our integration. Let us choose T_1. The integral thus becomes

$$\int_0^1 \int_0^u [(u^2 - v^2)^2 + u^2 v^2][2(u^2 + v^2)] dv du.$$